Topological Index

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The statement of the Aityah-Singer theorem that we are going towards is that two maps

$$K(TX) \to \mathbb{Z}$$

are equal. Our goal is to review some K theory and define carefully the topological index, one of the maps.

1 Background

A G-space is simply a topological space with an action of G on it. A G-vector bundle is just a G-space E which is also a vector bundle over a base B such that the action is compatible with the projection (commutes with acting on the base space as considered part of the total space, say via the zero section) and the map on the fibers by g is linear. Then $K_G(B)$ is just the group completion of the monoid of *complex* G-bundles over B under Whitney sum, as in the non-equivariant case. K_G theory is a *homotopy* invariant and is periodic. It moreover forms a ring under the tensor product of bundles.

Because we can always pull back vector bundles this makes K into a contravariant functor.

If X has a base point P then its reduced K theory is the kernel of the map on K theory induced by inclusion

$$K(X) \to K(P)$$

and it is denoted $\tilde{K}$. For a locally compact X we then define its K theory as the reduced K theory of the one point compactification. Note that this construction is simple but only functorial with respect to maps that behave well with the one point compactification. A sufficient condition is for the maps to be proper (preimage of compact is compact).

Remark. This is a rare case where the nlab page is actually quite good. In particular the proof of Bott periodicity there is actually illuminating and the steps used there are useful (and possibly referred to as Bott periodicity) in their own right.

Remark. You cannot pushforward vector (fiber) bundles as you will in general change the homotopy type of the fibers.

1.1 Thom Homomorphism

Let $V \to X$ be a *complex* vector bundle over a compact space. Then the K theory of the vector bundle is the reduced K theory of its compactification. Because X is compact we can think of the compactification of the total space as the so called Thom space of X. Indeed [Ati18] takes this as an equivalent definition. Thus

$$K(V) = \tilde{K}(V^+) = \tilde{K}(\mathrm{Th}(X))$$

by definition. By [Ati18, Cor 2.7.12] there is an isomorphism between the reduced K theory of the Thom space and the K theory of the base. Because $K(V)$ is a $K(X)$ module ($K(X)$ is a ring under tensor and K can be considered as a contravariant functor we get a ring map of $K(X) \to K(V)$ induced by the projection $V \to X$) we just need to define a certain Thom class, and then the isomorphism is given by multiplication with this class λ_V

$$K(X) \to K(V)$$

$$x \mapsto x.\lambda_V$$

This class is something like an analogue of an orientation for a generalised cohomology theory. The point is that it is a generator fiberwise.

Example. *What happens when $X = P$ is a point. Then $TP = P$. Consider an inclusion then of a point $P \to E$ into a real vector space E. Then we have that $TE \cong E \oplus E \cong E_{\mathbb{C}}$ and hence can be considered as a complex bundle over TP or P. Then the Thom isomorphism states that*

$$K(P) = K(TP) = K(TE)$$

The point is that a real bundle over a point gives a complex bundle by taking the tangent space, then we can employ the isomorphism.

Remark. Here are some informal remarks on the construction of this Thom class. First Aityah (in both paper and his book) gives an equivalent defintion of K theory in terms of *complexes* of vector bundles. In this setting the Thom class is simply the complex given by the exterior algebra of a vector bundle, $\Lambda^\bullet V \to V$.

It is non-trivial to track down a proof that these two notions of K-theory are the same, they reference an old paper by Aityah and Segal in [AS68] however I couldnt make it out in there clearly (although I didnt spend too long).

Our dear friend Jonah gave the following explination off the dome as it were: You start by considering the exterior algebra as just a vector bundle over the base X, we then pull it back along the bundle V

$$\begin{array}{ccc} \pi^*(\Lambda^V) & \longrightarrow & V \\ \downarrow & & \downarrow \\ \Lambda^V & \longrightarrow & X \end{array}$$

Because the exterior algebra breaks into its odd and even parts $\Lambda^{odd} V \oplus \Lambda^{even} V$ so too does the pullback. If we consider an element $\xi \in V_x$ then it acts on the exteror algebra of the fibers

$$\Lambda_x^V \to \Lambda_x^V$$

$$\omega \mapsto (\xi + \xi^*) \wedge \omega$$

This was not the exact map, the adjoint of ξ didnt act by directly wedging, or he wasnt sure, might have to dualise ω too etc. This then gives a well defined element of the relative K theory

$$(\Lambda^{odd} V, \varphi, \Lambda^{even} V) \in K(V, V - D)$$

where φ is some gluing of the disk bundle into $V - D$ and $K(V, V - D) \cong K(V)$. This is why I didnt go through this, I mean it should just be a vector bundle, cant we just describe it that way?

1.2 Functoriality

Given a map of spaces $f : M \to N$ we have already seen that we have a contravariantly induced map on K theory, what we construct here is a covaraiant map on the *tangent bundles* (derivations). Given an *embedding* of a compact space into a compact space $i : X \to Y$ then there is an induced map on the K_G theory $i!$ *of their tangent bundles*

$$i! : K_G(TX) \to K_G(TY)$$

This is still a functor into groups. The construction goes as follows. First we have a tubular neighborhood $X \to N \to Y$ that we may consider as the normal bundle of X in Y, call it ν. The tangent bundle of N is a tubular neighbourhood for TX in TY. Now we claim that we can describe TN as

$$\begin{array}{ccc} TN & \longrightarrow & TX \\ \downarrow & \lrcorner & \downarrow \\ N \otimes_{\mathbb{R}} \mathbb{C} & \longrightarrow & X \end{array}$$

Proof. Because $N_{\mathbb{C}} \cong N \oplus iN \cong N \oplus N$ it is sufficient to prove that TN is the pullback of $N \oplus N$ along the tangent bundle of X.

The pullback of $N \oplus N$ is given by $\{(n_1, n_2, v_x) \in N \oplus N \times TX : x = \pi_N(n_1, n_2)\}$, that is a tangent vector and two normal vectors at a given point. On the other hand $TN = \{(n, n') : n \in N, n' \in T_n N\}$. Lets examine n' more closely, it must be in $T_n N$, where $n = (x, n_x), n_x \in N_x$. Thus we are considering $T_{(x,n_x)}N$ which using a local trivialisation can be considered locally for $U \subseteq X$ open

$$T_{(x,n_x)}U \times N_x \cong T_x U \oplus T_{n_x} N_x \cong T_x U \oplus N_x$$

as N_x is just a vector space. Thus TN can be written out as

$$TN = \{(x, n_x, x', n'_x) : x \in X, x' \in T_x X, n_x, n'_x \in N_x\}$$

but now we can identify $(x, x') \in TX$ and see that

$$TN = \{(v_x, n_x, n'_x) : v_x \in TX, n_x, n'_x \in N_x\}$$

which is exactly $\pi^*(N \oplus N)$.

Now we have a complex vector bundle over TX and we can define

$$i! : K_G(TX) \xrightarrow{\varphi} K_G(TN) \xrightarrow{\iota} K_G(TY)$$

where φ is the Thom homomorphism and ι is the inclusion. [AS68] say that there is a canonical map from $K_G(TN) \xrightarrow{\iota} K_G(TY)$ induced by the inclusion because TN is *open* in TY. I no longer understand what they meant by this, what is the map?

Example. *Again consider the case of a point and a real vector bundle $E \to P$. Then a tubular neighbourhood of the point is given by all of E and we get that the induced map $K(P) = K(TP) \to K(TE)$ is just the Thom isomorphism.*

Remark. All of this is can be done such that the spaces are all G-equivariant.

1.3 Representation Theory

Lemma. *The K_G theory of a point is the representation ring of G.*

Proof. The definition of a G bundle in this context becomes just a vector space with a G action, in other words nothing but a *finite dimensional* representation of G. The monoid structure is direct sum. Then by definition the representation ring is the group completion of this structure (with product given by tensor)

Example. *If G is the trivial group then its representation ring is given by group completing the natural numbers, as a representation of the trivial group is just a vector space, which are determined by their dimension. Therefore the K theory of the point is just the integers.*

Example. *If $G = S^1$ then its representation ring is given by $\mathbb{Z}[x, x^{-1}]$.*

Example. *For a finite abelian group the representation ring is isomorphic to the group ring on the group of characters. This makes it clear that it will contain a copy of $\mathbb{Z}$.*

2 t-ind

Consider a compact differentiable G-manifold X, let $i : X \to E$ be a differentiable G-embedding of X in a *real* representation space of G (a real vector space on which G acts). Let $P \in E$ be the origin, then $j = i|_P : P \to E$ is the inclusion. We have induced maps on K_G theory then by

$$K_G(TX) \xrightarrow{i!} K_G(TE) \xleftarrow{j!} K_G(TP) = K_G(P) = R(G)$$

As we have previously noted though $j!$ is the Thom isomorphism and hence we have a well defined map

$$K_G(TX) \to R(G)$$

given by inverting $j!$ and composing.

Remark. Like everything we have stated the equivariant thing and just claimed that it works. In particular here it is not clear that there is such an embedding, although aparently their is (see the references in [AS68]).

3 A note on motivation and proof sketch

This theorem unifies three areas of mathematics, algebraic topology, analysis and representation theory. The analysis and algebraic topology are clear, the point is also to recognize that the *equivariant* K-theory is a global generalization of the representation ring of a group.

We do not have a deep understanding of the proof, but [AS68] provide a good summary of the proof and so we will riff on that.

3.1 Proof Sketch

First they provide some axioms for an "index function", $K_G(TX) \to R(G)$.

1. It is functorial in diffeomorphisms of X and homomorphisms of G.
2. For $X = P$ a point then an index function is the identity $R(G) \to R(G)$.
3. It commutes with the extension $i!$

$$\begin{array}{ccc} K(TX) & \xrightarrow{\quad i! \quad} & K(TY) \\ & {\scriptstyle ind_X} \searrow \quad \swarrow {\scriptstyle ind_Y} & \\ & R(G) & \end{array}$$

Topological index satisfies these axioms. And any index function satisfying them is the topological index.

The rest of the game is then to show that the analytic index satisfies the axioms. To show that the analytical index satisfies the axioms we perform two reductions. The first is to show that this set of axioms is equivalent to some other axioms that are easier to verify. These are along the lines of commuting with inclusions / pushforwards, certain other index functions are trivial and a form of multiplicativity (messy).

The second is that they show that if we have a operator on a compact X then they construct an operator *over the point* with the same index. This allows us to reduce to checking the axioms for operators over the point and of course that the construction is well behaved. Alternitively one can consider a inclusion $X \to Y$ where both spaces are compact. If F is an ellipitic operator on X, that is a linear function between the global sections of two bundles, then the main point is to extend this to a elliptic operator on Y, whilst preserving the index. If this can be done then since every compact manifold embedds into S^n for large n it would be sufficient to prove the index theorem for operators on spheres.

Left to do to understand the theorem:

- Describe the analytic index.
- Describe how to reduce to the case over the point.
- Examples: This is an analytic index, this is a topological index, they agree.
- See how different proofs go...

References

[AS68] M. F. Atiyah and I. M. Singer. The Index of Elliptic Operators: I. *Annals of Mathematics*, 87(3):484–530, 1968. Publisher: [Annals of Mathematics, Trustees of Princeton University on Behalf of the Annals of Mathematics, Mathematics Department, Princeton University].

[Ati18] Michael Francis Atiyah. *K-theory*. Advanced books classics. CRC Press, Taylor & Francis Group, Boca Raton Londo New York, 2018.